Domain wall solution of the Skyrme model

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Abstract

A class of domain-wall-like solutions of the Skyrme model is obtained analytically. They are described by the tangent hyperbolic function, which is a special limit of the Weierstrass $\wp$ function. The behavior of one of the two terms in the static energy density is like that of a domain wall. The other term in the static energy density does not vanish but becomes constant at the points far apart from the wall.

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I. INTRODUCTION

Skyrme Model [1] is an effective field theory describing hadrons [2], [3]. It is defined by the Lagrangian density

$$\mathcal{L}_S = -4c_2 \mathrm{tr}\left[(g^\dagger \partial_\mu g)(g^\dagger \partial^\mu g)\right] + \frac{c_4}{2}\mathrm{tr}\left([g^\dagger \partial_\mu g, g^\dagger \partial_\nu g][g^\dagger \partial^\mu g, g^\dagger \partial^\nu g]\right), \tag{1.1}$$

where $g(x)$ is an element of $SU(2)$ and c_2 and c_4 are coupling constants. If we define $A_\mu^\alpha(x)$ and $H_{\mu\nu}^\alpha(x)$ by

$$A_\mu^\alpha = \frac{1}{2i}\mathrm{tr}\left(\tau^\alpha g^\dagger \partial_\mu g\right), \tag{1.2}$$

$$H_{\mu\nu}^\alpha = \partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha, \tag{1.3}$$

$\mathcal{L}_S$ is expressed as

$$\mathcal{L}_S = 8c_2 A_\mu^\alpha A^{\alpha,\mu} - c_4 H_{\mu\nu}^\alpha H^{\alpha,\mu\nu}, \tag{1.4}$$

where $\tau_\alpha(\alpha = 1, 2, 3)$ are Pauli matrices. By definition, $A_\mu^\alpha(x)$ satisfies the condition

$$\partial_\mu A_\nu^\alpha - \partial_\nu A_\mu^\alpha = 2\varepsilon^{\alpha\beta\gamma} A_\mu^\beta A_\nu^\gamma. \tag{1.5}$$

The field equation is given as the conservation law

$$\partial_\mu J^{\alpha,\mu} = 0, \tag{1.6}$$

where $J^{\alpha,\mu}$ ($\alpha = 1, 2, 3$, $\mu = 0, 1, 2, 3$) are defined by

$$J^{\alpha,\mu} = 2c_2 A^{\alpha,\mu} + c_4 \varepsilon^{\alpha\beta\gamma} H^{\beta,\mu\nu} A_\nu^\gamma. \tag{1.7}$$

The field $J^{\alpha,\mu}(x)$ is proportional to the isospin current of the model. Another important current of the Skyrme model is the baryon number current $N^\lambda(x)$ [1] defined by

$$N^\lambda = \frac{1}{12\pi^2}\varepsilon^{\lambda\mu\nu\rho}\varepsilon^{\alpha\beta\gamma} A_\mu^\alpha A_\nu^\beta A_\rho^\gamma. \tag{1.8}$$

The conservation law $\partial_\lambda N^\lambda = 0$ follows solely from the definition of $N^\lambda(x)$ irrespective of the field equation for $A_\mu^\alpha(x)$. It was shown that solitons of the Skyrme model possess polyhedral structures [4], [5]. As for the analytic solutions for these models, only a few simple examples [1, 6, 7] are known. Skyrme [1] found that the configuration

$$g(x) = h(k \cdot x) \tag{1.9}$$

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with $k \cdot x = k_\mu x^\mu$ leading to $A_\mu^\alpha(x) = k_\mu f^\alpha(k \cdot x)$ satisfies the field equation if k_μ is light-like: $k^2 = 0$. For these solutions, the field $H_{\mu\nu}^\alpha(x)$ vanishes and $g(x)$ is independent of the coupling constants c_2 and c_4. In a recent paper [6], Yamashita and the present authors obtained a solution of the form

$$g(x) = h(k \cdot x, \ l \cdot x, \ m \cdot x), \tag{1.10}$$

with k, l, m being three momenta satisfying $k^2 = l^2 = m^2 = 0$. In that case, the field $A_\mu^\alpha(x)$ is expressed as

$$A_\mu^\alpha(x) = \frac{k_\mu}{\kappa^1} a^\alpha(\omega, \omega') + \frac{l_\mu}{\kappa^2} b^\alpha(\omega, \omega') + \frac{m_\mu}{\kappa^3} c^\alpha(\omega, \omega'), \tag{1.11}$$

where the variables ω and ω' are linear in x^μ and κ^i is defined as

$$\kappa^i = \sqrt{\frac{c_4}{c_2} \frac{(k^i \cdot k^j)(k^i \cdot k^k)}{(k^j \cdot k^k)}}, \tag{1.12}$$

with the triplet (i, j, k) being (1,2,3) or (2,3,1) or (3,1,2) and $k^1 = k$, $k^2 = l$, $k^3 = m$. In this case, we can see that the solutions dependent on c_2, c_4 nontrivially, and that the field $H_{\mu\nu}^\alpha(x)$ and the baryon number density are nonvanishing. Under the Ansatz used in ref. [6], the quantities $|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and $|\boldsymbol{c}|$ are constants and $\boldsymbol{a} \cdot \boldsymbol{b}$, $\boldsymbol{b} \cdot \boldsymbol{c}$ and $\boldsymbol{c} \cdot \boldsymbol{a}$ are described with the help of the function

$$
\begin{aligned}
K(\omega) &= \wp(\omega + \omega_3) \\
&= e_3 + \frac{(e_3 - e_1)(e_3 - e_2)}{\wp(\omega) - e_3} \\
&= e_3 + (e_2 - e_3)\mathrm{sn}^2\left(\sqrt{e_1 - e_3}\,\omega, \ \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}\right),
\end{aligned} \tag{1.13}
$$

where $\wp(z)$ is the Weierstrass $\wp$ function satisfying the differential equation

$$[\wp'(z)]^2 = 4\left[\wp(z) - e_1\right]\left[\wp(z) - e_2\right]\left[\wp(z) - e_3\right]. \tag{1.14}$$

Here the constants e_1, e_2 and e_3 are complicated functions of $|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and $|\boldsymbol{c}|$. They are real and are assumed to satisfy $e_1 > e_2 > e_3$. $\mathrm{sn}(u, k)$ is the Jacobi elliptic function of u with the modulus k, $2\omega_3$ is the second fundamental period of $\wp(z)$, and $\omega = L \cdot x$ is a linear combination of $k \cdot x$, $l \cdot x$ and $m \cdot x$. We note that L^2 is equal to c_2/c_4 multiplied by a constant independent of the momenta k, l and m. The function $K(\omega)$ oscillates even for large $|\omega|$. Noting that the static energy density and the baryon number density are described by $K(\omega)$ and its derivative, respectively, we see that they also oscillate at the spatial infinity.

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In this paper, we consider the limit that the function $\mathrm{sn}(u, k)$ appearing in Eq. (1.13) tends to $\mathrm{Tanh}(u)$. Then the simple behavior of $\mathrm{Tanh}(u)$ suggests that we might obtain a domain wall configuration in the limit. This limit is realized if the parameter e_1 tends to e_2. Because of the complicated dependency of (e_1, e_2, e_3) on $|\boldsymbol{a}|$, $|\boldsymbol{b}|$ and $|\boldsymbol{c}|$, it is necessary to show that there indeed exists a set $(|\boldsymbol{a}|, |\boldsymbol{b}|, |\boldsymbol{c}|)$ realizing $e_1 \to e_2$ and $K(\omega) \to e_3 + (e_2 - e_3) \left[\mathrm{Tanh}(\sqrt{e_2 - e_3}\omega) \right]^2$.

As will be shown in later sections, this is the case. As is seen from Eq. (1.1), the static energy density of the Skyrme model consists of two terms: the one quadratic and the other quartic in field variables. It turns out that the behavior of the quartic term is like that of a domain wall. It approaches to zero at points far apart from the wall. The behavior of the quadratic term is also like that of a domain wall. It approaches to a constant at points far apart from the wall. It should be noted that the last constant is non-vanishing The baryon number density concentrates near the wall. The total baryon number, however, vanishes.

This paper is organized as follows. In II, we briefly introduce the solutions of the Skyrme model described by the $\wp$ function. In III, we explain how to explore domain wall solutions. the detail of the analysis is given. IV is devoted to a summary and discussions.

II.   SOLUTIONS DESCRIBED BY $\wp$ FUNCTION

A.   Field equation

To be self-contained, we briefly review the method of ref. [6] in this section. Introducing ξ^i by

$$\xi^i = \frac{k^i \cdot x}{\kappa^i} \quad (i = 1, 2, 3) \tag{2.1}$$

and write $\xi = \xi^1$, $\eta = \xi^2$ and $\zeta = \xi^3$, we find that the integrability condition (1.5) yields

$$\begin{aligned}
\frac{\partial \boldsymbol{a}}{\partial \eta} - \frac{\partial \boldsymbol{b}}{\partial \xi} &= 2(\boldsymbol{b} \times \boldsymbol{a}), \\
\frac{\partial \boldsymbol{a}}{\partial \zeta} - \frac{\partial \boldsymbol{c}}{\partial \xi} &= 2(\boldsymbol{c} \times \boldsymbol{a}), \\
\frac{\partial \boldsymbol{b}}{\partial \zeta} - \frac{\partial \boldsymbol{c}}{\partial \eta} &= 2(\boldsymbol{c} \times \boldsymbol{b}).
\end{aligned} \tag{2.2}$$

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Here, $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are three-dimensional vectors introduced in Eq.(1.11). If we use ξ, η, and ζ, the field equation (1.6) becomes

$$\left(\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\zeta}\right)(\boldsymbol{a} + \boldsymbol{D}) + \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\zeta}\right)(\boldsymbol{b} + \boldsymbol{E}) + \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)(\boldsymbol{c} + \boldsymbol{F}) = 0, \qquad (2.3)$$

where $\boldsymbol{D}$, $\boldsymbol{E}$, $\boldsymbol{F}$ are given by

$$\boldsymbol{D} = \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{a}) + \boldsymbol{a} \times (\boldsymbol{c} \times \boldsymbol{a}) + \boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{a}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a}),$$

$$\boldsymbol{E} = \boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{b}) + \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b}) + \boldsymbol{a} \times (\boldsymbol{c} \times \boldsymbol{b}), \qquad (2.4)$$

$$\boldsymbol{F} = \boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{c}) + \boldsymbol{c} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) + \boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{c}).$$

To solve this highly nonlinear field equation, we must introduce some Ansätze.

B.   Ansätze

The Ansätze adopted in ref. [6] is summarized as

$$\begin{pmatrix} \frac{\partial\boldsymbol{a}}{\partial\xi} & \frac{\partial\boldsymbol{a}}{\partial\eta} & \frac{\partial\boldsymbol{a}}{\partial\zeta} \\ \frac{\partial\boldsymbol{b}}{\partial\xi} & \frac{\partial\boldsymbol{b}}{\partial\eta} & \frac{\partial\boldsymbol{b}}{\partial\zeta} \\ \frac{\partial\boldsymbol{c}}{\partial\xi} & \frac{\partial\boldsymbol{c}}{\partial\eta} & \frac{\partial\boldsymbol{c}}{\partial\zeta} \end{pmatrix} = \begin{pmatrix} \lambda & \gamma & 0 \\ \gamma + 2 & -\rho & 0 \\ 0 & 0 & 0 \end{pmatrix}(\boldsymbol{a} \times \boldsymbol{b}) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & \alpha \\ 0 & \alpha + 2 & -\nu \end{pmatrix}(\boldsymbol{b} \times \boldsymbol{c})$$

$$+ \begin{pmatrix} -\kappa & 0 & \beta + 2 \\ 0 & 0 & 0 \\ \beta & 0 & \mu \end{pmatrix}(\boldsymbol{c} \times \boldsymbol{a}), \qquad (2.5)$$

where $\alpha, \beta, \gamma, \kappa, \lambda, \mu, \nu, \rho$ and σ are constants. It was found in ref. [6] that the Ansätze is consistent with the integrability condition (2.2) and the field equation (2.3) if the parameters satisfy

$$\rho = \frac{\mu}{\nu}\frac{\alpha(\alpha + 2)}{\alpha + \beta + 2}, \quad \lambda = -\frac{\nu}{\mu}\frac{\beta(\beta + 2)}{\alpha + \beta + 2}, \quad \gamma = -\frac{(\alpha + 2)(\beta + 2)}{\alpha + \beta + 2},$$

$$\kappa = -\frac{\beta(\beta + 2)}{\mu}, \quad \sigma = -\frac{\alpha(\alpha + 2)}{\nu}. \qquad (2.6)$$

Therefore we can regard the four constants α, β, μ and ν as independent.

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C. Solution of the Ansätze

From the above Ansätze, we see that $\boldsymbol{a}^2$, $\boldsymbol{b}^2$ and $\boldsymbol{c}^2$ are constants. We also obtain

$$\boldsymbol{a} \cdot \boldsymbol{b} = (\alpha + \beta + 2)J(\omega) + d_1, \tag{2.7}$$

$$\boldsymbol{b} \cdot \boldsymbol{c} = \mu J(\omega) + d_2, \tag{2.8}$$

$$\boldsymbol{c} \cdot \boldsymbol{a} = -\nu J(\omega) + d_3, \tag{2.9}$$

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = -\alpha(\alpha + 2)\frac{\mu}{\nu}\frac{dJ(\omega)}{d\omega}, \tag{2.10}$$

$$\omega = \rho\kappa\xi - \rho\sigma\eta + \mu\sigma\zeta, \tag{2.11}$$

where $J(\omega)$ is a function of ω and d_1, d_2 and d_3 are arbitrary constants. With the help of the identity

$$[\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})]^2 = \begin{vmatrix} \boldsymbol{a}^2 & (\boldsymbol{a} \cdot \boldsymbol{b}) & (\boldsymbol{a} \cdot \boldsymbol{c}) \\ (\boldsymbol{b} \cdot \boldsymbol{a}) & \boldsymbol{b}^2 & (\boldsymbol{b} \cdot \boldsymbol{c}) \\ (\boldsymbol{c} \cdot \boldsymbol{a}) & (\boldsymbol{c} \cdot \boldsymbol{b}) & \boldsymbol{c}^2 \end{vmatrix}, \tag{2.12}$$

it is concluded that $J(\omega)$ is related to the function $K(\omega)$ in Eq. (1.13) by

$$K(\omega) = \frac{1}{z_1}J(\omega) - \frac{z_2}{z_1}, \tag{2.13}$$

$$z_1 = -\frac{2\alpha^2(\alpha + 2)^2\mu}{(\alpha + \beta + 2)\nu^3}, \tag{2.14}$$

$$z_2 = \frac{-\mu^2\boldsymbol{a}^2 - \nu^2\boldsymbol{b}^2 - (\alpha + \beta + 2)^2\boldsymbol{c}^2 - 2\mu\nu d_1 + 2(\alpha + \beta + 2)(\mu d_3 - \nu d_2)}{6(\alpha + \beta + 2)\mu\nu}. \tag{2.15}$$

As is seen from (1.13) and (1.14), $K(\omega)$ satisfies

$$\left[\frac{dK(\omega)}{d\omega}\right]^2 = 4\left[K(\omega)\right]^3 - g_2 K(\omega) - g_3$$

$$= 4\left[K(\omega) - e_1\right]\left[K(\omega) - e_2\right]\left[K(\omega) - e_3\right], \tag{2.16}$$

where $g_2, g_3, e_1, e_2,$ and e_3 are real constants satisfying $e_1 \geq e_2 \geq e_3$. They are related by

$$e_1 + e_2 + e_3 = 0, \quad e_2e_3 + e_3e_1 + e_1e_2 = -\frac{1}{4}g_2, \quad e_1e_2e_3 = \frac{1}{4}g_3. \tag{2.17}$$

If we set $K(\omega) = \wp(\omega + \text{const}) = \wp(\omega + \omega_3)$ with $\omega_3 = \frac{i}{2}\int_{-\infty}^{e_3} \frac{du}{\sqrt{(e_1 - u)(e_2 - u)(e_3 - u)}}$, we find that $K(\omega)$ is regular for real values of ω and expressed by the Jacobi elliptic function by (1.13).

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D. Solution of the field equation

The field equation is now reduced to six algebraic equations. Three of them are solved by fixing d_1, d_2, d_3 as

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \frac{1}{2ABC} \begin{pmatrix} -C(As_0 + Bt_0 - Cu_0) \\ A(As_0 - Bt_0 - Cu_0) \\ -B(As_0 - Bt_0 + Cu_0) \end{pmatrix}, \tag{2.18}$$

$$A = (\alpha + \beta + \gamma) - 3\alpha + (\nu - \mu) + (\rho - \sigma), \tag{2.19}$$

$$B = (\alpha + \beta + \gamma) - 3\beta + (\kappa - \lambda) + (\nu - \mu), \tag{2.20}$$

$$C = (\alpha + \beta + \gamma) - 3\gamma + (\rho - \sigma) + (\kappa - \lambda), \tag{2.21}$$

$$s_0 = (\beta + \gamma + 2 + \rho - \mu)\boldsymbol{a}^2 - \nu\boldsymbol{b}^2 + \sigma\boldsymbol{c}^2 + 2(\alpha + 1), \tag{2.22}$$

$$t_0 = \mu\boldsymbol{a}^2 + (\alpha + \gamma + 2 + \nu - \lambda)\boldsymbol{b}^2 - \kappa\boldsymbol{c}^2 + 2(\beta + 1), \tag{2.23}$$

$$u_0 = -\rho\boldsymbol{a}^2 + \lambda\boldsymbol{b}^2 + (\alpha + \beta + 2 + \kappa - \sigma)\boldsymbol{c}^2 + 2(\gamma + 1). \tag{2.24}$$

The other three equations are reduced to the restrictions on the parameters $(\alpha, \beta, \mu, \nu)$. They can be written in the form $p_i\mu^2 + 2h_i\mu\nu + q_i\nu^2 + 2g_i + 2f_i\nu + r_i = 0$ $(i = 1, 2, 3)$, where $p_i, q_i, , r_i, f_i, g_i$ and h_i are polynomials of α and β. They were solved in ref. [6]. Two examples of the allowed sets are given by

$$(\alpha = \beta, \mu, \nu) = \left(-2 - \frac{1}{\sqrt{2}} + \frac{\sqrt{5 + 4\sqrt{2}}}{2}, 1, -1\right) \tag{2.25}$$

and

$$(\alpha = \beta, \mu, \nu) = \left(-2 - \frac{1}{\sqrt{2}} - \frac{\sqrt{5 + 4\sqrt{2}}}{2}, 1, -1\right). \tag{2.26}$$

III. EXPLORING DOMAIN WALL SOLUTIONS

A. Conditions to realize a domain wall

Recalling that $K(\omega)$ contains the factor $(e_2 - e_3)\left[\text{sn}\left(\sqrt{e_1 - e_3}\,\omega, \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}\right)\right]^2$, we here consider some limiting cases of this factor. From the definition

$$u = \int_0^{\text{sn}(u,k)} \frac{dz}{\sqrt{(1 - z^2)(1 - kz^2)}} \tag{3.1}$$

of the Jacobi elliptic function, we easily find

$$\text{sn}(u, 0) = \sin u, \quad \text{sn}(u, 1) = \tanh u. \tag{3.2}$$

The sn $\to$ sin limit in our case is realized by $e_2 \to e_3$ and leads to trivial $K(\omega) : K(\omega) = e_3$. On the other hand, the sn $\to$ tanh limit is realized by

$$e_1 \to e_2, \quad e_3 \to -2e_2 \tag{3.3}$$

and leads to nontrivial $K(\omega)$:

$$K(\omega) = -2e_2 + 3e_2[\tanh \chi]^2, \quad \chi = \sqrt{3e_2}\,\omega. \tag{3.4}$$

From the property of the function $[\tanh \chi]^2$, we expect that a domain wall-like structure might appear in this limit. Since e_1, e_2 and e_3 depend on the parameters α, β, μ, ν and on the constants a^2, b^2, c^2 in a complicated manner, we have to check weather this limit can be realized or not.

B.   A special case

We consider the case

$$\mu = 1, \quad \nu = -1, \quad \alpha = \beta = -2 - \frac{1}{\sqrt{2}} + \frac{\sqrt{5 + 4\sqrt{2}}}{2} = -0.398768 \cdots \tag{3.5}$$

and assume that $\boldsymbol{k}$, $\boldsymbol{l}$, $\boldsymbol{m}$ are orthogonal to each other. The conditions to be considered are summarized as

(1) $e_1 = e_2, \quad e_3 = -2e_2$

$$\to g_2 = 12(e_2)^2, \quad g_3 = -8(e_2)^3. \tag{3.6}$$

$$\to g_2^3 = 27g_3^2,$$

(2) $e_2 > 0 \to g_3 < 0.$ \qquad (3.7)

(3) $a^2 b^2 \geq (a \cdot b)^2, \quad b^2 c^2 \geq (b \cdot c)^2, \quad c^2 a^2 \geq (c \cdot a)^2$ for any χ. \qquad (3.8)

We note that $(a \cdot b)$, $(b \cdot c)$, $(c \cdot a)$ are linear in a^2, b^2, c^2 and contain $[\tanh \chi]^2$, g_2 is a polynomial of a^2, b^2, c^2 of second order, g_3 is a polynomial of a^2, b^2, c^2 of third order, $g_2^3 - 27g_3^2$ is a polynomial of a^2, b^2, c^2 of sixth order.

C. Static energy density

The energy momentum tensor is given by

$$T_{\mu\nu} = \frac{\partial \mathcal{L}_s}{\partial A^{\alpha,\mu}} A^{\alpha}_{\nu} - \eta_{\mu\nu} \mathcal{L}_s. \tag{3.9}$$

Setting

$$k = (k_0, k_0, 0, 0), \quad l = (l_0, 0, l_0, 0), \quad m = (m_0, 0, 0, m_0), \quad k_0, l_0, m_0 > 0, \tag{3.10}$$

we have

$$\kappa^1 = \sqrt{\frac{c_4}{c_2}} k_0, \quad \kappa^2 = \sqrt{\frac{c_4}{c_2}} l_0, \quad \kappa^3 = \sqrt{\frac{c_4}{c_2}} m_0 \tag{3.11}$$

and

$$(\boldsymbol{A}_0, \boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{A}_3) = \frac{c_2}{c_4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}). \tag{3.12}$$

The variable ω defined in Eq. (2.11) is more explicitly given by

$$\omega = L \cdot x, \tag{3.13}$$

$$L_\mu = \frac{\beta(\beta+2)}{2\nu^2(\beta+1)}[-\nu\beta(\beta+2)\frac{k_\mu}{\kappa^1} + \mu\beta(\beta+2)\frac{l_\mu}{\kappa^2} - 2\mu\nu(\beta+1)\frac{m_\mu}{\kappa^3}], \tag{3.14}$$

We note that ω depends on the time variable x_0.

In the case $\mu = 1$, $\nu = -1$, we have

$$L = (L_0, \boldsymbol{L}), \tag{3.15}$$

$$L_0 = 2\Lambda(\beta^2 + 3\beta + 1), \tag{3.16}$$

$$\boldsymbol{L} = \Lambda\varpi\hat{\boldsymbol{u}}, \tag{3.17}$$

$$\hat{\boldsymbol{u}} = \frac{1}{\sigma}\big(\beta(\beta+2), \beta(\beta+2), 2(\beta+1)\big), \tag{3.18}$$

$$\Lambda = \frac{\beta(\beta+2)}{2(\beta+1)}\sqrt{\frac{c_2}{c_4}}, \tag{3.19}$$

$$\varpi = \sqrt{2\beta^2(\beta+2)^2 + 4(\beta+1)^2}, \tag{3.20}$$

$$L^2 = L^\mu L_\mu = 2\beta(\beta+2)(\beta^2 + 6\beta + 4)\Lambda^2. \tag{3.21}$$

Since L^2 for $\beta = -2 - \frac{1}{\sqrt{2}} + \frac{\sqrt{5+4\sqrt{2}}}{2}$ is negative, there exists a Lorentz transformation $(L_0, \boldsymbol{L}) \rightarrow (0, \boldsymbol{L}')$. Since $\omega = L \cdot x = L' \cdot x'$ does not contain the time variable x_0' in the new system, $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ can now be regarded as static. For selfcontainedness, we describe the above Lorentz transformation in some detail in the Appendix.

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The static energy density E is now given as follows. It consists of two parts E_1 and E_2 :

$$E = \frac{8c_2{}^2}{c_4}[E_1 + E_2], \tag{3.22}$$

$$E_1 = 2\Big\{ (v_1^2 \boldsymbol{a}^2 + v_2^2 \boldsymbol{b}^2 + v_3^2 \boldsymbol{c}^2)$$

$$+ (2v_1 v_2 - 1)\boldsymbol{a} \cdot \boldsymbol{b} + (2v_2 v_3 - 1)\boldsymbol{b} \cdot \boldsymbol{c} + (2v_3 v_1 - 1)\boldsymbol{c} \cdot \boldsymbol{a} \Big\} \tag{3.23}$$

$$E_2 = (4v_2 v_3 - 1)\boldsymbol{A}^2 + (4v_3 v_1 - 1)\boldsymbol{B}^2 + (4v_1 v_2 - 1)\boldsymbol{C}^2$$

$$+ (2 + 4v_3^2 - 4v_2 v_3 - 4v_3 v_1)\boldsymbol{A} \cdot \boldsymbol{B} + (2 + 4v_1^2 - 4v_3 v_1 - 4v_1 v_2)\boldsymbol{B} \cdot \boldsymbol{C}$$

$$+ (2 + 4v_2^2 - 4v_1 v_2 - 4v_2 v_3)\boldsymbol{C} \cdot \boldsymbol{A}, \tag{3.24}$$

where $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, v_1, v_2$, and v_3 are defined by

$$\boldsymbol{A} = \boldsymbol{b} \times \boldsymbol{c}, \quad \boldsymbol{B} = \boldsymbol{c} \times \boldsymbol{a}, \quad \boldsymbol{C} = \boldsymbol{a} \times \boldsymbol{b}, \tag{3.25}$$

$$v_1 = v_2 = \cosh\theta + \frac{\beta(\beta+1)}{\varpi}\sinh\theta, \quad v_3 = \cosh\theta + \frac{2(\beta+1)}{\varpi}\sinh\theta, \tag{3.26}$$

$$\sinh\theta = -\sqrt{2}\frac{\beta^2 + 3\beta + 1}{\sqrt{-\beta(\beta+2)(\beta^2 + 6\beta + 4)}}, \tag{3.27}$$

$$\cosh\theta = \sqrt{-\frac{\beta^4 + 4\beta^3 + 6\beta^2 + 4\beta + 2}{\beta(\beta+2)(\beta^2 + 6\beta + 4)}}. \tag{3.28}$$

D.   Explicit solution

To obtain a set of constants $(\boldsymbol{a}^2, \boldsymbol{b}^2, \boldsymbol{c}^2)$ realizing the three conditions at the top of this section, we adopt a simplifying assumption $\boldsymbol{a}^2 = \boldsymbol{b}^2 \equiv x$. If we set $\boldsymbol{c}^2 \equiv y$, we find that g_2 and g_3 are polynomials of x, y of second and third order, respectively. It can be seen that many terms in g_2 and g_3 vanish when J defined by

$$J = \beta^4 + 8\beta^3 + 18\beta^2 + 16\beta + 4. \tag{3.29}$$

vanishes. We here note that $\beta = -2 - \frac{1}{\sqrt{2}} + \frac{\sqrt{5+4\sqrt{2}}}{2}$ is one of the solutions of $J = 0$. Then, the condition $(g_2)^3 - 27(g_3)^2 = 0$ simplifies to

$$y(p_0 x + p_1 y)(q_0 x^2 + 8q_1 xy + 8q_2 y^2)^2 = 0, \tag{3.30}$$

$$p_0 = -74 - 52\sqrt{2} + 22\sqrt{5 + 4\sqrt{2}} + 16\sqrt{10 + 8\sqrt{2}}, \tag{3.31}$$

$$p_1 = -599 - 424\sqrt{2} + 183\sqrt{5 + 4\sqrt{2}} + 130\sqrt{10 + 8\sqrt{2}}, \tag{3.32}$$

$$q_0 = -175979 - 124436\sqrt{2} + 53907\sqrt{5 + 4\sqrt{2}} + 38118\sqrt{10 + 8\sqrt{2}}, \tag{3.33}$$

$$q_1 = -292919 - 207125\sqrt{2} + 89729\sqrt{5 + 4\sqrt{2}} + 63448\sqrt{10 + 8\sqrt{2}}, \tag{3.34}$$

$$q_2 = -272444 - 192647\sqrt{2} + 83457\sqrt{5 + 4\sqrt{2}} + 59013\sqrt{10 + 8\sqrt{2}}. \tag{3.35}$$

The only solution of Eq.(3.30) compatible with the conditions $x > 0, y > 0, g_2 > 0$ and $g_3 < 0$ is

$$y = \tau x, \tag{3.36}$$

$$\tau = \frac{C + \sqrt{D}}{F} = 0.724872\cdots, \tag{3.37}$$

where C, D and F are given by

$$C = -585838 - 414250\sqrt{2} + 179458\sqrt{5 + 4\sqrt{2}} + 126896\sqrt{10 + 8\sqrt{2}}, \tag{3.38}$$

$$D = 726708054834 + 535418248054\sqrt{2} - 303040027330\sqrt{5 + 4\sqrt{2}} \tag{3.39}$$

$$- 214281658296\sqrt{10 + 8\sqrt{2}} + 32820149302\sqrt{114 + 80\sqrt{2}}, \tag{3.40}$$

$$F = 4\left(-272444 - 192647\sqrt{2} + 83457\sqrt{5 + 4\sqrt{2}} + 59013\sqrt{10 + 8\sqrt{2}}\right). \tag{3.41}$$

Utilizing the above results and together with formulae (2.7), (2.8) and (2.9), condition (3.8) is satisfied automatically. Now we turn to the static energy density E. As is seen from (3.24), the second part E_2 of E consists only of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. Under the assumptions adopted here, it turns out that all of $\lim_{|\chi|\to\infty} \boldsymbol{A}$, $\lim_{|\chi|\to\infty} \boldsymbol{B}$, $\lim_{|\chi|\to\infty} \boldsymbol{C}$ are proportional to $q_0 x^2 + 8q_1 xy + 8q_2 y^2$. We then see that, under the choice $y = \tau x$, E_2 vanishes at points far apart from the plane $\chi = \boldsymbol{L}' \cdot \boldsymbol{x} = 0$. This behavior is just like that of a domain wall. On the other hand, the first part E_1 of E tends to a non-vanishing constant as χ tends to ∞. We show the behavior of $\boldsymbol{A}^2$, $\boldsymbol{B}^2 = \boldsymbol{C}^2$ in Figs. 1 and 2. The behavior of the static energy densities E_1 and E_2 are shown in Figs. 3 and 4. The behavior of total static energy density

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$E = E_1 + E_2$ is shown in Fig.5. We note that these figures are drawn under the assumption $x = \sqrt{y}$ in addition to (3.36), yielding $x = \tau, y = \tau^2$. The baryon number density in this case is given by

$$N_0(x) = N\left(\frac{c_2}{c_4}\right)^{3/2},\tag{3.42}$$

where N and n_0 are given by

$$N = n_0 \frac{dK(\omega)}{d\omega},\tag{3.43}$$

$$n_0 = \frac{1}{2\pi^2}\frac{\beta^3(\beta+2)^3}{\beta+1}.\tag{3.44}$$

The behavior of baryon number density is depicted in Fig. 6. We find that it is positive on one side of the plane and negative on the other side and the integral $\int_{-\infty}^{\infty} d\omega N_0(x)$ vanishes. We note that, for instance in the $x = \sqrt{y}$ case, we have $\int_{-\infty}^{0} d\omega N(x) = -\int_{0}^{\infty} d\omega N(x) = 0.0297476$.

We also note that another solution similar to the above one can be obtained also in the case of (2.26). In this case, we have $\int_{-\infty}^{0} d\omega N(x) = -\int_{0}^{\infty} d\omega N(x) = 0.0151418$ for $x = \tau, y = \tau^2$.
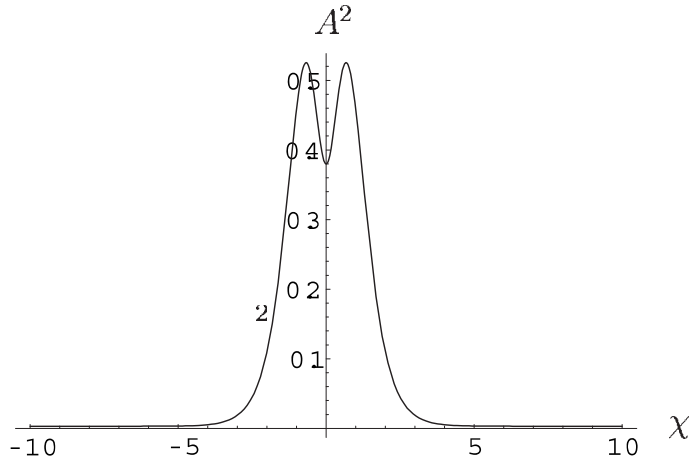
FIG. 1: Behavior of $\boldsymbol{A}^2$ ($a^2 = b^2 = 0.729476\cdots$, $c^2 = 0.532136\cdots$)

IV. SUMMARY

We have explored the domain wall solution of the Skyrme model. To obtain it, we have considered a limiting case of the previously obtained solution of the Skyrme model
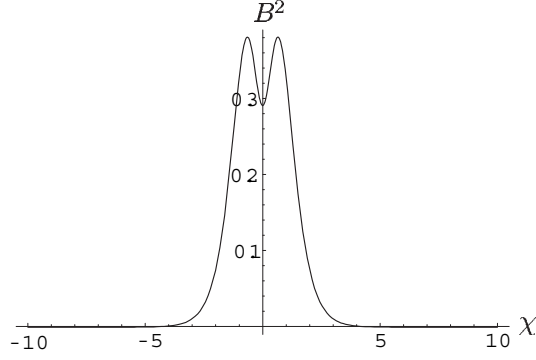
FIG. 2: Behavior of $\boldsymbol{B}^2 = \boldsymbol{C}^2$ ($a^2 = b^2 = 0.729476\cdots, c^2 = 0.532136\cdots$)
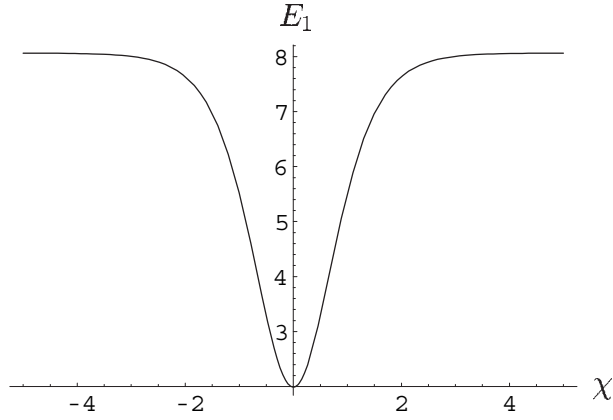

FIG. 3: Static energy densities E_1

described by the Weierstrass $\wp$ function. In the limit considered, the function $\mathrm{sn}(u, k)$ tends to $\mathrm{sn}(u, 1) = \tanh(u)$ leading to a static domain wall-like solution of the Skyrme model. Because of the complicated structure of the model, we needed to show explicitly that the above limit is indeed realizable. We have shown that there indeed exists a set of parameters which realize the limit. The two terms constituting the static energy density of the Skyrme model were also investigated. The behavior of the term quartic in the field variables turns out just like that of a domain wall : it is non-vanishing only in the neighborhood of a plane(wall) in the space. On the other hand, the term quadratic in the field variables approaches to a non-vanishing constant at points far apart from the wall. The baryon number density concentrates near the plane. It is positive in one side of the plane and negative on the other side. The total baryon number vanishes and the topological stability of the solution is not maintained. To obtain solutions with non-vanishing baryon number, it would be necessary to obtain solutions expressed not by $\tanh^2(u)$ but by. for example, $\tanh(u)$.
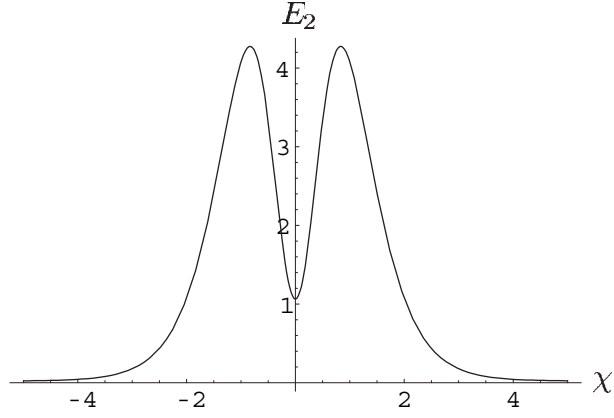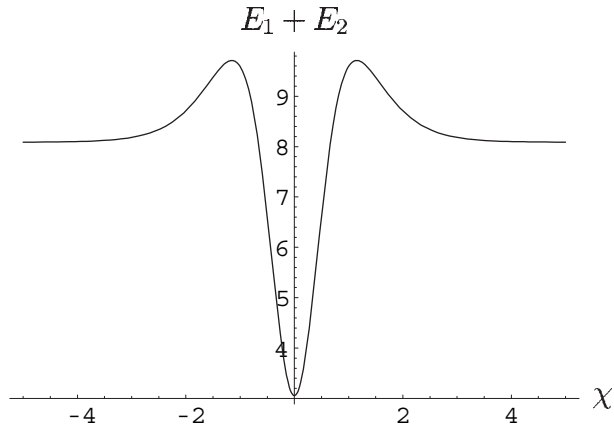
FIG. 4: Static energy densities E_2
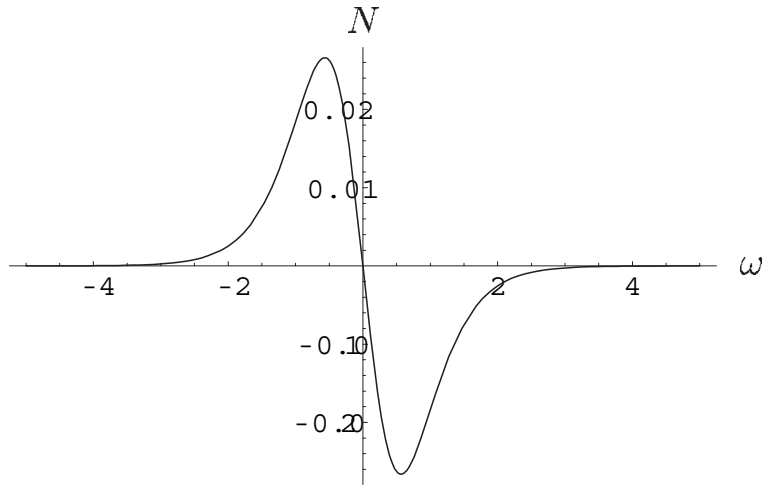

FIG. 5: Total static energy

FIG. 6: Baryon number density

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Acknowledgments

APPENDIX A: DETAIL OF LORENTZ TRANSFORMATION

The energy momentum tensor (3.9) is rewritten as

$$
T_{\mu\nu} = \frac{16c_2{}^2}{c_4}\Bigg[k_{\mu\nu}\boldsymbol{a}^2 + l_{\mu\nu}\boldsymbol{b}^2 + m_{\mu\nu}\boldsymbol{c}^2 + r_{\mu\nu}(\boldsymbol{a}\cdot\boldsymbol{b}) + s_{\mu\nu}(\boldsymbol{b}\cdot\boldsymbol{c})
$$
$$
+ u_{\mu\nu}(\boldsymbol{c}\cdot\boldsymbol{a}) - \eta_{\mu\nu}(\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{c} + \boldsymbol{c}\cdot\boldsymbol{a}) \Bigg]
$$
$$
+ \frac{8c_2{}^2}{c_4}\Bigg[4(k_{\mu\nu}\boldsymbol{B}\cdot\boldsymbol{C} + l_{\mu\nu}\boldsymbol{C}\cdot\boldsymbol{A} + m_{\mu\nu}\boldsymbol{A}\cdot\boldsymbol{B})
$$
$$
+ 2r_{\mu\nu}(\boldsymbol{C}^2 - \boldsymbol{C}\cdot\boldsymbol{A} - \boldsymbol{B}\cdot\boldsymbol{C})
$$
$$
+ 2s_{\mu\nu}(\boldsymbol{A}^2 - \boldsymbol{C}\cdot\boldsymbol{A} - \boldsymbol{A}\cdot\boldsymbol{B})
$$
$$
+ 2u_{\mu\nu}(\boldsymbol{B}^2 - \boldsymbol{B}\cdot\boldsymbol{C} - \boldsymbol{A}\cdot\boldsymbol{B})
$$
$$
- \eta_{\mu\nu}(\boldsymbol{A}^2 + \boldsymbol{B}^2 + \boldsymbol{C}^2 - 2\boldsymbol{A}\cdot\boldsymbol{B} - 2\boldsymbol{B}\cdot\boldsymbol{C} - 2\boldsymbol{C}\cdot\boldsymbol{A}) \Bigg], \qquad (A1)
$$

where the following matrices have been made use of:

$$
(k_{\mu\nu}) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
(l_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
(m_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
(r_{\mu\nu}) = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad
(s_{\mu\nu}) = \begin{pmatrix} 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad
(u_{\mu\nu}) = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}. \qquad (A2)
$$

We here describe some details of the Lorentz transformation from the system with $L = (L_0, \boldsymbol{L}) = (L_0, -|\boldsymbol{L}|\hat{\boldsymbol{u}})$ to that with $L' = (0, \boldsymbol{L}') = (0, \varepsilon\sqrt{-L^2})$. It is given by

$$\begin{pmatrix} 0 \\ -\sqrt{-L^2} \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} L_0 \\ -|\boldsymbol{L}| \end{pmatrix}, \tag{A3}$$

from which we obtain

$$\sinh\theta = \frac{l_0}{\sqrt{-L^2}} = -\sqrt{2}\frac{\beta^2 + 3\beta + 1}{\sqrt{-\beta(\beta+2)(\beta^2 + 6\beta + 4)}}. \tag{A4}$$

With the help of the matrices

$$R_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A5}$$

and

$$R_2 = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}, \tag{A6}$$

we find

$$R_2 R_1{}^t\hat{\boldsymbol{u}} = \frac{1}{\varpi} \begin{pmatrix} \sqrt{2}\beta(\alpha+2)\cos\phi + 2(\beta+1)\sin\phi \\ 0 \\ -\sqrt{2}\beta(\beta+2)\sin\phi + 2(\beta+1)\cos\phi \end{pmatrix}. \tag{A7}$$

If we set $\cos\phi = \frac{\sqrt{2}\beta(\beta+2)}{\varpi}$, we have

$$R_2 R_1{}^t\hat{\boldsymbol{u}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{A8}$$

So in this coordinate system, $\boldsymbol{L}$ is given by

$$^t\boldsymbol{L} = \begin{pmatrix} \Lambda\varpi \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -|\boldsymbol{L}| \\ 0 \\ 0 \end{pmatrix}. \tag{A9}$$

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Combining the above two transformations, we find that the static energy density is given by Eq. (3.22).

_____

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